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Methods of Design Sensitivity Analysis in Structural Optimization

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Design sensitivity analysis plays a central role in structural optimization, since virtually all optimization methods require computation of derivatives of structural response quantities with respect to design variables. Three fundamentally different approaches to design sensitivity analysis are presented. These have been used extensively in the structural optimization literature. They are the virtual load method, the state space method, and the design space method. An analysis of these methods indicates that the state space and design space methods are more general than the virtual load method. Moreover, the virtual load method, when applicable, is a special case of the state space method. Any one of these procedures may be incorporated into an optimality criterion or a mathematical programming method for structural optimization.

Nomenclature

b = $(m \times 1)$ design variable vector
C^ℓ = $(n \times m)$ matrix defined in Eq. (13)
$e_{ij\ell}$ = virtual strain energy density stored in the ith group of members due to the jth virtual load in going through the displacements of the ℓth loading condition
$F^\ell(b)$ = $(n \times 1)$ vector representing ℓth effective loading condition for the structure
k^i = stiffness matrix for the ith element
$\bar{k}^i$ = $(n \times n)$ "blown-up" stiffness matrix for the ith element
$K(b)$ = $(n \times n)$ structural stiffness matrix
ℓ = index identifying loading conditions
L_i = length or surface area of members connected to the ith design variable
n = number of degrees of freedom for the finite element representation of the structure
n_c = number of loading condition
n_d = number of displacement constraints
n_s = number of stress constraints
n_v = number of constraint violations at an optimal design iteration
$n(\ell)$ = total number of constraints for the ℓth loading condition
$p^{i\ell}$ = vector of nodal forces for the ith element under the ℓth loading condition
q^j = $(n \times 1)$ virtual displacement vector for the jth constraint
$Q^j(b)$ = $(n \times 1)$ virtual load vector related to the jth constraint
$u_j^\ell(b)$ = a limit value for the jth constraint under the ℓth loading condition
U = upper Cholesky factor of the stiffness matrix K
z^ℓ = $(n \times 1)$ nodal displacement vector under the ℓth loading condition for the finite-element representation of the structure
z_j^ℓ = jth component of z^ℓ
$\bar{z}_j$ = allowable displacement for the jth component
β^i = a Boolean matrix such that $\beta^i z^\ell$ is the nodal displacement vector for the ith element under the ℓth loading condition
δb = a vector of small changes in b
δz^ℓ = a vector of small changes in z^ℓ
σ_j^ℓ = stress at the jth point or the jth member under the ℓth loading condition
$\bar{\sigma}_j$ = allowable jth stress component
ν = design iteration number
ψ_j^ℓ = jth constraint function for the ℓth loading condition
$\Lambda^{j\ell}$ = $(m \times 1)$ design derivative vector of the jth constraint function for the ℓth loading condition

Matrix Calculus Notation

$d\psi_j^\ell/db$ = $(1 \times m)$ row vector; the total derivative of $\psi_j^\ell = \psi_j^\ell(b, z^\ell(b))$ with respect to b
$\partial z^\ell/\partial b$ = $(n \times m)$ matrix
$(\partial/\partial b)[K(b)q^j]$ = $(n \times m)$ matrix; q^j is not differentiated with respect to b in this calculation
$\partial K(b)/\partial b_i$ = $(n \times n)$ matrix
$\partial u_j^\ell(b)/\partial b$ = $(1 \times m)$ row vector
$(\partial/\partial b)[K(b)z^\ell]$ = $(n \times m)$ matrix; z^ℓ is not differentiated with respect to b in this calculation

I. Introduction

DURING the last few years, there has been considerable research activity in the area of computational methods for optimization of structural systems. The literature on the subject is so voluminous that it is difficult to cite all available references, short of making this a general review paper. Venkayya, Berke, and co-workers,[1-7] Schmit and co-workers,[8-12] Haug, Arora, and co-workers,[13-19] and others[20-23] have made significant contributions in advancing the state-of-art of structural optimization. Recent books of Gallagher and Zienkiewicz,[24] Pope and Schmit,[25] and the review article of Moses[26] summarize the state-of-art in the field. The objective of structural optimization is to find design variables for the system that minimize a cost function and satisfy various performance requirements. The design

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variables usually sought are cross-sectional areas, widths, thicknesses, and other dimensions of structural members. Let these design variables be given by an m-vector b.

Two philosophically different approaches to structural optimization have been followed in the literature. The first technique is called the numerical optimality criterion method, or indirect method. Most recent optimality criteria methods are based on solving the nonlinear set of equations obtained by applying the Kuhn-Tucker necessary conditions to the original design problem. These equations are solved numerically in an iterative manner. Traditionally, the following recurrence formula for calculating optimum values of design variables has been used in these methods:

$$b_j^{(\nu+1)} = c_j^{(\nu)} b_j^{(\nu)} \qquad j=1,2,\ldots,m; \quad \nu=0,1,2,\ldots \tag{1}$$

where $c_j^{(\nu)}$ is a multiplier, ν is the iteration number, and $b^{(0)}$ is a starting design. Venkayya, Berke, and co-workers[1-7] have used this approach extensively in the optimal design of a variety of structures. Other researchers[20-22] have also developed optimization procedures based on this concept. Recently, Khot et al.[7] studied various optimality criteria methods and presented relationships between them.

The second basic approach is called the mathematical programming method, and is more direct in nature. In this approach, one starts with a design estimate b and finds a search direction based on the local behavior of cost and constraint functions. A small move in this direction gives an improved design. Thus, a sequence of design modifications is generated that reduces the cost function, simultaneously satisfying design constraints. Traditionally, the following iterative formula has been used for design improvement in these methods:

$$b^{(\nu+1)} = b^{(\nu)} + \delta b^{(\nu)} \qquad \nu=0,1,2,\ldots \tag{2}$$

where $\delta b^{(\nu)}$ is a small change in design at the νth iteration. There are several procedures for calculating the optimum design change $\delta b^{(\nu)}$ at each iteration. Haug, Arora, and co-workers[13-19] and others[23,27] have used the gradient projection method extensively. Schmit and co-workers[8-12] have developed several algorithms based on the sequential unconstrained minimization techniques (SUMT) and applied them to several classes of structural systems.

It has been shown that Eqs. (1) and (2) are completely interchangeable. Based on this idea, Arora and Haug[28] recently analyzed optimality criteria and the gradient projection methods. Relations between the two approaches are presented and hybrid methods for structural optimization are suggested.

In all the foregoing optimization algorithms one needs derivatives of structural response quantities, such as stress, displacement, buckling load, and natural frequency, with respect to the design variables. In mathematical programming methods, these derivatives are used in calculating the design change vector $\delta b^{(\nu)}$, whereas in optimality criteria methods they are used in calculating the multipliers, $c_j^{(\nu)}$. This design derivative (design sensitivity analysis) portion of any structural optimization algorithm constitutes a major segment of the total calculations. Thus it is important to carry out the design sensitivity analysis as efficiently as possible if the algorithm is to be applied to large practical structures such as aircraft, civil engineering, and other special-purpose complex structures.

It may be noted that design derivatives are also important in their own right. They represent trends that are important to the designer in changing his design estimate. Therefore, the designer may use this information directly in interactive computer-aided design procedures.

The purpose of this paper is to present three fundamentally different approaches to design sensitivity analysis in structural optimization, and to analyze the relationship between them. Design sensitivity analysis of an eigenvalue related to a buckling load or natural frequency is well documented.[13,29] The procedure can be integrated into any optimization algorithm. This problem is, therefore, not addressed in this paper. Rather, design sensitivity analysis for the stress, displacement, and other constraints is presented.

Finally, a comment about the notation used in this paper is in order. A standard matrix and vector notation is used throughout the presentation. Superscripts are used to represent different vectors and matrices; subscripts are used to represent components of matrices and vectors.

II. Structural Analysis

Most practical structures must be modeled by the finite-element method to calculate their response to applied loads. In discussing design sensitivity analysis procedures, one must therefore adopt this structural analysis method. In terms of displacements, the equilibrium equation for a finite-element model of a structure is given as[30,31]

$$K(b)z^\ell = F^\ell(b) \qquad \ell=1,2,\ldots,n_c \tag{3}$$

The variable z^ℓ describes a state of the structural system and is therefore called a state variable. Equation (3) is also called a state equation. The structural stiffness matrix $K(b)$ is synthesized from stiffness matrices of each of the finite elements of the structure.[30,31] The external load vector F^ℓ will depend on design variables if body forces or thermal effects are included in the analysis.

For a given design, Eq. (3) is a set of linear equations that determine nodal coordinate vectors z^ℓ. The stiffness matrix $K(b)$ is symmetric and usually banded. Efficient numerical procedures are used to decompose K as:

$$K = U^T U \tag{4}$$

where U is an upper triangular Cholesky matrix. Forward and backward substitutions are then used to solve the z^ℓ from Eq. (3). Once nodal displacements of the structure are known, each finite element of the structure is considered as an isolated free body and element nodal forces are obtained as:

$$p^{i\ell} = k^i \beta^i z^\ell \tag{5}$$

From nodal forces of an element, element stresses are easily computed.

III. Methods of Design Sensitivity Analysis

The stress constraint at the jth critical point or for jth member of the structure under the ℓth loading condition is expressed as:

$$\sigma_j^\ell - \bar{\sigma}_j \le 0 \qquad j=1,2,\ldots,n_s, \quad \ell=1,2,\ldots,n_c \tag{6}$$

where n_s is the number of stress constraints, n_c is the number of loading conditions, and $\bar{\sigma}_j$ is an allowable stress. The stress σ_j^ℓ is computed using the member-end forces obtained from Eq. (5) and the loads applied to the member. It should be noted that σ_j^ℓ is an explicit fucntion of b and z^ℓ.

The displacement constraint for the jth degree of freedom under the ℓth loading condition is:

$$z_j^\ell - \bar{z}_j \le 0 \qquad j=1,2,\ldots,n_d, \quad \ell=1,2,\ldots,n_c \tag{7}$$

where n_d is the number of displacement constraints, z_j^ℓ is a computed displacement, and $\bar{z}_j$ is an allowable displacement.

The problem of design sensitivity analysis is to compute derivatives of constraint Eqs. (6) and (7) with respect to design variables. For the presentation that follows, it is convenient to treat constraints of Eqs. (6) and (7) as special cases of the more general functional constraint

$$\psi_j^\ell(b,z^\ell) \le 0 \qquad j=1,2,\ldots,n(\ell), \quad \ell=1,2,\ldots,n_c \tag{8}$$

where $n(\ell)$ is the total number of constraints for the ℓth loading condition. Equation (8) can be used to represent stress, displacement, or other constraints that depend on b and z^ℓ.

A. Virtual Load Method

It is noted in Ref. 5 that the virtual load procedure of design sensitivity analysis was first used by Barnett and Hermann[32] for statically determinate trusses with a single displacement constraint. The procedure was later extended for statically indeterminate structures with multiple deflection constraints.[3,4] This procedure, coupled with a numerical optimality criterion method, has been used by Venkayya, Berke, and co-workers[1-7] for a number of years. It can also be integrated into mathematical programming methods.[33,34] Here, a more general form of the virtual load method of design sensitivity analysis is first presented. Design derivatives expressions used by Venkayya, Berke, and co-workers,[1-7] and Sander and co-workers[33,34] are then derived from the general procedure.

The virtual load procedure is applicable if the jth constraint of Eq. (8) for the ℓth loading condition is expressed as:

$$\psi_j^\ell(b,z^\ell) \equiv Q^{j^T}(b)z^\ell - u_j^\ell(b) \le 0 \tag{9}$$

where $Q^j(b)$ is an $(n\times 1)$ vector and $u_j^\ell(b)$ is a constraint bound. The vector Q^j in Eq. (9) is taken to be a function of the design variable vector b, which is required for bending and shear stress constraints for the beam and plate-type elements. For these elements, Q^j is calculated from Eq. (5). In the case of stress constraints for truss and constant strain elements[18,33] and for the nodal displacement constraints, the vector Q^j may not depend on the design variable vector b. For example, for the jth displacement constraint of Eq. (7), Q^j has unit value at the jth location and zeroes elsewhere. Also, the constraint bound u_j^ℓ in Eq. (9) may depend on the design variable vector b when a buckling constraint for beam-type elements is considered or when thermal forces and body forces are considered in the analysis. Therefore, to maintain a degree of generality in the virtual load procedure, Q^j and u_j^ℓ are treated as functions of the design variable vector in the following derivation.

Now one defines a vector q^j $(n\times 1)$ to be the solution of the equation

$$K(b)q^j = Q^j(b) \tag{10}$$

Note that the previous decomposition of K is available from Eq. (4) for use in Eq. (10). Comparing Eqs. (3) and (10), one may interpret Q^j as the jth virtual load vector and q^j as the corresponding jth virtual displacement vector. The quantity $Q^{j^T} z^\ell$ in Eq. (9) is then the virtual work done by the jth virtual load in going through the displacements of the ℓth loading condtion. This quantity is sometimes referred to as flexibility and the constraint of Eq. (9) is called a flexibility constraint.[7,33]

Substituting Eq. (10) into Eq. (9) and using the symmetry of $K(b)$, one obtains

$$\psi_j^\ell(b,z^\ell) \equiv q^{j^T}K(b)z^\ell - u_j^\ell(b) \tag{11}$$

Taking the total derivative of both sides of Eq. (11) with respect to the design variable vector b, using the fact that z^ℓ and q^j depend on b, and taking transpose, one obtains

$$\frac{\mathrm{d}\psi_j^{\ell^T}(b,z^\ell(b))}{\mathrm{d}b} \equiv \Lambda^{j\ell} = \left[\frac{\partial q^{j^T}}{\partial b}K(b)z^\ell + q^{j^T}\frac{\partial}{\partial b}\{K(b)z^\ell\} + q^{j^T}K(b)\frac{\partial z^\ell}{\partial b} - \frac{\partial u_j^\ell(b)}{\partial b}\right]^T \tag{12}$$

Here $\Lambda^{j\ell}$ is the required design derivative vector of dimension $(m\times 1)$ for the constraint of Eq. (9). For an explanation of the vector calculus notation, refer to the Nomenclature at the beginning of the paper.

Taking the total derivative of both sides of Eqs. (3) and (10) with respect to the design variable b, using the fact that z^ℓ and q^j depend on b, and rearranging,

$$K(b)\frac{\partial z^\ell}{\partial b} = C^\ell \equiv \frac{\partial F^\ell(b)}{\partial b} - \frac{\partial}{\partial b}\{K(b)z^\ell\} \tag{13}$$

$$K(b)\frac{\partial q^j}{\partial b} = \frac{\partial Q^j(b)}{\partial b} - \frac{\partial}{\partial b}\{K(b)q^j\} \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (12), one obtains

$$\Lambda^{j\ell} = \left[z^{\ell^T}\frac{\partial Q^j(b)}{\partial b} - z^{\ell^T}\frac{\partial}{\partial b}\{K(b)q^j\} + \frac{\partial}{\partial b}\{K(b)z^\ell\}^T q^j + C^{\ell^T}q^j - \frac{\partial u_j^{\ell^T}(b)}{\partial b}\right] \tag{15}$$

Note that in Eq. (15)

$$\frac{\partial}{\partial b}\{K(b)z^\ell\}^T q^j = z^{\ell^T}\frac{\partial}{\partial b}\{K(b)q^j\} \tag{16}$$

because z^ℓ and q^j are treated as constants during partial differentation. Therefore, the design derivative vector obtained by the virtual load method is given as:

$$\Lambda^{j\ell} = \left[z^{\ell^T}\frac{\partial Q^j(b)}{\partial b} + C^{\ell^T}q^j - \frac{\partial u_j^{\ell^T}(b)}{\partial b}\right] \tag{17}$$

It is instructive to derive the design-derivative expressions used by Venkayya, Berke, and co-workers[1-7] and Sander and co-workers[33,34] from the general expression given in Eq. (17). In these references, Q^j, F^ℓ, and u_j^ℓ are taken to be independent of b, which is the case in some structural design applications. Therefore, $\partial Q^j/\partial b = \partial F^\ell/\partial b = \partial u_j^\ell/\partial b = 0$ in Eq. (17). The ith component of the design derivatives vector from Eq. (17) is now

$$\Lambda_i^{j\ell} \equiv -\frac{\partial}{\partial b_i}\{K(b)z^\ell\}^T q^j = -z^{\ell^T}\frac{\partial K(b)}{\partial b_i}q^j \tag{18}$$

Also, K is taken as a linear function of b_i, as is the case in some structural design applications,[1-7,33,34] so $\partial K(b)/\partial b_i = (1/b_i)\bar{k}^i$ where $\bar{k}^i$ is a "blown-up" element stiffness matrix for members connected to the ith design variable b_i. Thus, Eq. (18) becomes

$$\Lambda_i^{j\ell} = -e_{ij\ell}L_i \qquad e_{ij\ell} = (z^{\ell^T}\bar{k}^i q^j)/(b_iL_i) \tag{19}$$

where $e_{ij\ell}$ may be viewed as a virtual strain energy density and (b_iL_i) is the volume of members connected to the ith design variable.[7,33] Equation (19) represents exactly the expression used in Refs. 1-7, 33, and 34.

B. State Space Method

The state space approach has been developed by Haug, Arora, and co-workers.[13-19] In this method, state variable vector z^ℓ is first treated as an independent variable. An adjoint relationship is then introduced to express the effect of a variation in z^ℓ in terms of the variation in design variable vector b.[13] Hence, the method is called a state space method. In contrast to the virtual load procedure, no special functional form of constraints in Eq. (8) is assumed and a variational approach is followed in deriving the design derivative vector.

A first variation of the function $\psi_j^\ell(b,z^\ell)$, treating b and z^ℓ as independent variables, is given as:

$$\delta\psi_j^\ell(b,z^\ell)=\frac{\partial\psi_j^\ell(b,z^\ell)}{\partial b}\delta b+\frac{\partial\psi_j^\ell(b,z^\ell)}{\partial z^\ell}\delta z^\ell \tag{20}$$

where $\delta\psi_j^\ell$ is a first-order change in the function ψ_j^ℓ, δb is a small change in b, and δz^ℓ is the corresponding small change in z^ℓ. Partial derivatives in Eq. (20) are computed at the given value of b and the computed value of z^ℓ. In the state space method, one expresses the second term in Eq. (20) as a function of δb, such that the equation may be written as:

$$\delta\psi_j^\ell=\Lambda^{j\ell^T}\delta b \tag{21}$$

where $\Lambda^{j\ell}$ is the $(m\times 1)$ column vector of design derivatives, that is,

$$\Lambda_i^{j\ell}=\frac{\mathrm{d}\psi_j^\ell(b,z(b))}{\mathrm{d}b_i}$$

In order to achieve the objective of eliminating the second term from Eq. (20), one first defines an identity[13] by premultiplying Eq. (3) by the transpose of an $(n\times 1)$ adjoint variable vector $\lambda^{j\ell}$ that is associated with the constraint function ψ_j^ℓ. Taking the first variation of both sides of the resulting equation and rearranging, one obtains

$$\lambda^{j\ell^T}K(b)\delta z^\ell=\lambda^{j\ell^T}C^\ell\delta b \tag{22}$$

where the matrix C^ℓ is given in Eq. (13). One now defines $\lambda^{j\ell}$ as the solution of

$$K(b)\lambda^{j\ell^T}=\frac{\partial\psi_j^\ell(b,z^\ell)}{\partial z^\ell}^T \tag{23}$$

so Eq. (22) becomes

$$\frac{\partial\psi_j^\ell(b,z^\ell)}{\partial z^\ell}\delta z^\ell=\lambda^{j\ell^T}C^\ell\delta b \tag{24}$$

Substituting from Eq. (24) into Eq. (20) and comparing the result with Eq. (21), one obtains the following expression for $\Lambda^{j\ell}$ by the state space method:

$$\Lambda^{j\ell}=\frac{\partial\psi_j^{\ell^T}(b,z)}{\partial b}+C^{\ell^T}\lambda^{j\ell} \tag{25}$$

The adjoint variable $\lambda^{j\ell}$ needed in Eq. (25) is efficiently obtained from the adjoint equation Eq. (23), using previously calculated factors of $K(b)$ in Eq. (4).

C. Design Space Method

The design space approach to design sensitivity analysis was first suggested by Fox,[35] and has been used by several authors.[9-12] In this approach, the state variable is assumed to be given as $z^\ell=z^\ell(b)$. Then $\delta z^\ell=(\partial z^\ell/\partial b)\delta b$ is substituted in Eq. (20). Comparing the resulting expression with Eq. (21), one obtains the design derivative vector as:

$$\Lambda^{j\ell}=\left[\frac{\partial\psi_j^\ell(b,z)}{\partial b}+\frac{\partial\psi_j^\ell(b,z)}{\partial z^\ell}\frac{\partial z^\ell}{\partial b}\right]^T \tag{26}$$

The matrix $\partial z^\ell/\partial b$ required in Eq. (26) is computed from Eq. (13). Note that Eq. (13) has the same coefficient matrix as in Eq. (3). Therefore, the decomposed form of $K(b)$ is available and only forward and backward substitutions are needed to solve for the matrix $\partial z^\ell/\partial b$. Thus, one obtains a very concise derivation and formula for $\Lambda^{j\ell}$.

IV. Analysis of Methods

The three design sensitivity analysis procedures presented in Sec. III yield the same design derivatives for stress, displacement, or any other constraint. However, it is now shown that substantially different amounts of computational effort may be required.

In structural optimization, only the active constraints at any iteration require design sensitivity calculations. Let n_v be the total number of active constraints at a design iteration. The virtual load procedure then requires calculation of n_v virtual displacement vectors from Eq. (10), and the state space method requires calculation of n_v adjoint vectors from Eq. (23). The computational effort for these calculations is identical for the two approaches and the use of Eqs. (17) and (25) completes their respective design sensitivity analyses. However, the virtual load method is slightly restricted in the sense that constraints must be expressed in the form of Eq. (9).

It is now shown that the virtual load method of design sensitivity analysis, when applicable, can be derived from either the state space method or the design space method. For this purpose, one assumes that as in the virtual load method the functional form of the constraint is given as in Eq. (9). For the constraint of Eq. (9), the adjoint equation, Eq. (23), is:

$$K(b)\lambda^{j\ell}=Q^j(b) \tag{27}$$

Comparing Eqs. (10) and (27), one observes that the adjoint vector $\lambda^{j\ell}$ is identical to the virtual displacement vector q^j. For Eq. (9)

$$\frac{\partial\psi_j^\ell(b,z^\ell)}{\partial b}=\frac{\partial Q^{j^T}(b)}{\partial b}z^\ell-\frac{\partial u_j^\ell(b)}{\partial b} \tag{28}$$

Substituting Eq. (28) into the formula for design derivatives by the state space method [Eq. (25)], one obtains the formula of Eq. (17) for design derivatives by the virtual load procedure.

Substituting for $\partial\psi_j^\ell(b,z^\ell)/\partial b$ from Eq. (28), $\partial z^\ell/\partial b$ from Eq. (13) and $\partial\psi_j^\ell(b,z^\ell)/\partial z^\ell$ into Eq. (26) for design derivatives by the design space method, one obtains

$$\Lambda^{j\ell}=\left[z^{\ell^T}\frac{\partial Q^j(b)}{\partial b}-\frac{\partial u_j^{\ell^T}(b)}{\partial b}+C^{\ell^T}K^{-1}Q^j(b)\right] \tag{29}$$

Noting from Eq. (10) that $K^{-1}Q^j(b)=q^j$, one observes that Eq. (29) is identical to Eq. (17).

The difference between the design space and state space methods of design sensitivity analysis is the way in which the term related to δz^ℓ is treated in Eq. (20). In the state space approach, Eq. (24) and solution of the adjoint variable $\lambda^{j\ell}$ from Eq. (23) completes the sensitivity analysis. In the design space analysis, $\partial z^\ell/\partial b$ is solved from Eq. (13) for use in Eq. (26). Thus, the major difference between the state space and the design space approaches is the difference in the number of vectors that must be determined from Eqs. (23) and (13), respectively. In the state space approach this number is n_v and for the design space approach it is mn_c. Therefore, depending on the values of n_v and mn_c, one method is to be preferred over the other. It is noted, however, that near the optimum point n_v is always $\leq m$. Further, in iterative optimization one generally begins with a near-feasible design, so throughout the iterative optimization process, one has $n_v < m$. Thus, for practical purposes, one always has $n_v \leq m$,[13] so if $n_c > 1$, $n_v \ll mn_c$. Thus the state space method is more efficient than the design space method, often by factors up to ten.

V. Conclusions

Three different approaches to design sensitivity analysis that have been used extensively in the structural optimization literature are presented and analyzed. Based on this study, the following conclusions are drawn:

1) The state space and design space methods of design sensitivity analysis are more general than the virtual load method.

2) Whenever the virtual load method is applicable, it generates a sequence of operations for design derivative calculation that is identical to the one generated by the state space method.

3) Since one generally has $n_v < mn_c$, the state space method (or the virtual method when applicable) for design derivative calculation is superior to the design space method.

4) Any one of the three procedures of design sensitivity analysis may be integrated into an optimality criterion or a mathematical programming method for structural optimization. However, due to generality and efficiency, the state space method should be preferred over the other two methods.

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